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Nonstandard $\mathrm{GL}_h(n)$ quantum groups and contraction of covariant q-bosonic algebras*

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Abstract

$\mathrm{GL}_h(n) \times \mathrm{GL}_h(m)$-covariant h-bosonic algebras are built by contracting the $\mathrm{GL}_q(n) \times \mathrm{GL}_q(m)$-covariant q-bosonic algebras considered by the present author some years ago. Their defining relations are written in terms of the corresponding R_h-matrices. Whenever $n = 2$, and $m = 1$ or 2, it is proved by using $\mathrm{U}_h(\mathrm{sl}(2))$ Clebsch-Gordan coefficients that they can also be expressed in terms of coupled commutators in a way entirely similar to the classical case. Some $\mathrm{U}_h(\mathrm{sl}(2))$ rank-1/2 irreducible tensor operators, recently contructed by Aizawa in terms of standard bosonic operators, are shown to provide a realization of the h-bosonic algebra corresponding to $n = 2$ and $m = 1$.

1 Introduction

It is well known that the Lie group GL(2) admits, up to isomorphism, only two quantum group deformations with central determinant: the standard deformation $\mathrm{GL}_q(2)$, and the Jordanian deformation $\mathrm{GL}_h(2)$ [1]. The quantum group $\mathrm{GL}_h(2)$, or $\mathrm{SL}_h(2)$, and the dual quantum algebra of the latter, $\mathrm{U}_h(\mathrm{sl}(2))$ [2], have been the subject of many recent investigations, among which one may quote the determination of the $\mathrm{U}_h(\mathrm{sl}(2))$ universal $\mathcal{R}$-matrix [3].

Two useful tools have been devised for the Jordanian deformation study. One of them is a contraction procedure that allows one to construct the latter from the standard deformation [4]. In other words, $\mathrm{GL}_h(2)$ can be obtained from $\mathrm{GL}_q(2)$ by a singular limit

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of a similarity transformation. Such a technique has been generalized by Alishahiha to higher-dimensional quantum groups [5].

The other tool is a nonlinear invertible map between the generators of $U_h(sl(2))$ and sl(2) [6], yielding an explicit and simple method for constructing the finite-dimensional irreducible representations (irreps) of $U_h(sl(2))$. In addition, it has provided an explicit formula for $U_h(sl(2))$ Clebsch-Gordan coefficients (CGC) [7], as well as bosonic or fermionic realizations of irreducible tensor operators (ITO) for $U_h(sl(2))$ [8].

The purpose of the present communication is to apply the contraction procedure of Ref. [4], as generalized by Alishahiha [5], to the $GL_q(n) \times GL_q(m)$-covariant q-bosonic algebras constructed by the present author some years ago [9], and recently rederived by Fiore by another procedure [10]. As a result, we will obtain $GL_h(n) \times GL_h(m)$-covariant h-bosonic algebras. We will then consider the cases where $n = 2$, $m = 1$, and $n = m = 2$ in more detail, and establish some relations with the works of Aizawa on ITO [8], and of Van der Jeugt on CGC for $U_h(sl(2))$ [7].

2 Contraction of $GL_q(N)$

The quantum group $GL_q(N)$ is defined by the RTT-relations, $R'T'_1T'_2 = T'_2T'_1R'$, where $T' = \left(T'_{ij}\right) \in GL_q(N)$, $T'_1 = T' \otimes I$, $T'_2 = I \otimes T'$, and

$$R' = R'_q = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \left(q - q^{-1}\right) \sum_{i<j} e_{ij} \otimes e_{ji}, \qquad (1)$$

with i, j running over 1, 2, ..., N, and e_{ij} denoting the $N \times N$ matrix with entry 1 in row i and column j, and zeros everywhere else. An equivalent form of the RTT-relations is obtained by replacing $R' = R'_{12}$ by R'^{-1}_{21}. Throughout this communication, q-deformed objects will be denoted by primed quantities, whereas unprimed ones will represent h-deformed objects.

Let us consider the similarity transformation $R'' = \left(g^{-1} \otimes g^{-1}\right) R'(g \otimes g)$, $T'' = g^{-1}T'g$, where g is the $N \times N$ matrix defined by $g = \sum_i e_{ii} + \eta e_{1N}$, in terms of some parameter $\eta = h/(q-1)$ [4, 5]. The RTT-relations simply become $R''T''_1T''_2 = T''_2T''_1R''$.

Whenever q goes to 1, although η becomes singular, the latter have a definite limit $RT_1T_2 = T_2T_1R$, where $T = \lim_{q\to 1} T''$, and

$$\begin{aligned} R &= R_h = \lim_{q \to 1} R'' \\ &= \sum_{ij} e_{ii} \otimes e_{jj} + h\Big[e_{11} \otimes e_{1N} - e_{1N} \otimes e_{11} + e_{1N} \otimes e_{NN} - e_{NN} \otimes e_{1N} \\ &\quad + 2 \sum_{i=2}^{N-1} \left(e_{1i} \otimes e_{iN} - e_{iN} \otimes e_{1i}\right)\Big] + h^2 e_{1N} \otimes e_{1N}. \end{aligned} \qquad (2)$$

The resulting R-matrix is triangular, i.e., it is quasitriangular and $R_{12}^{-1} = R_{21}$, showing that the two equivalent forms of RTT-relations for $GL_q(N)$ have actually the same contraction limit. The matrix elements T_{ij} generate $GL_h(N)$.

3 $\mathrm{GL}_q(n) \times \mathrm{GL}_q(m)$-covariant q-bosonic algebras

Let us consider two different copies of $\mathrm{GL}_q(N)$, corresponding to possibly different dimensions n, m, and let us denote quantities referring to $\mathrm{GL}_q(n)$ by ordinary letters (R', T', ...), and quantities referring to $\mathrm{GL}_q(m)$ by script ones ($\mathcal{R}'$, $\mathcal{T}'$, ...). The elements T'_{ij}, i, $j = 1$, 2, ... n, of $\mathrm{GL}_q(n)$, and $\mathcal{T}'_{st}$, s, $t = 1$, 2, ... m, of $\mathrm{GL}_q(m)$ are assumed to commute with one another.

In Ref. [9], q-bosonic creation and annihilation operators $\boldsymbol{A}'^{+}_{is}$, $\tilde{\boldsymbol{A}}'_{is}$, $i = 1, 2, \ldots n$, $s = 1, 2, \ldots m$, that are double ITO of rank $[1\dot{0}]_n[1\dot{0}]_m$, and $[\dot{0}-1]_n[\dot{0}-1]_m$ with respect to $\mathrm{U}_q(gl(n)) \times \mathrm{U}_q(gl(m))$, respectively, were constructed in terms of standard q-bosonic operators [11] a'^{+}_{is}, a'_{is}, $i = 1, 2, \ldots, n$, $s = 1, 2, \ldots, m$, acting in a tensor product Fock space $F = \prod_{i=1}^{n} \prod_{s=1}^{m} F_{is}$. The annihilation operators $\boldsymbol{A}'_{is}$ contragredient to $\boldsymbol{A}'^{+}_{is}$ were also considered. Both sets of annihilation operators $\tilde{\boldsymbol{A}}'_{is}$ and $\boldsymbol{A}'_{is}$, $i = 1, 2, \ldots, n$, $s = 1, 2, \ldots, m$, are related through the equation $\tilde{\boldsymbol{A}}' = \boldsymbol{A}'\boldsymbol{C}'$, where $\boldsymbol{C}' = C'\mathcal{C}'$, $C' = \sum_i (-1)^{n-i} q^{-(n-2i+1)/2} e_{ii'}$, and $\mathcal{C}' = \sum_s (-1)^{m-s} q^{-(m-2s+1)/2} e_{ss'}$, with $i' = n - i + 1$, $s' = m - s + 1$.

The operators $\boldsymbol{A}'^{+}_{is}$, $\boldsymbol{A}'_{is}$, or $\boldsymbol{A}'^{+}_{is}$, $\tilde{\boldsymbol{A}}'_{is}$, generate with $\boldsymbol{I} = I\mathcal{I}$ a $\mathrm{U}_q(\mathrm{gl}(n)) \times \mathrm{U}_q(\mathrm{gl}(m))$-module algebra or $\mathrm{GL}_q(n) \times \mathrm{GL}_q(m)$-comodule algebra, whose q-commutation relations can be compactly written in coupled form by using $\mathrm{U}_q(\mathrm{gl}(n)) \times \mathrm{U}_q(\mathrm{gl}(m))$ CGC. When rewritten in componentwise form, such relations can be expressed in terms of the $\mathrm{GL}_q(n)$ and $\mathrm{GL}_q(m)$ R-matrices as [9]

$$\begin{aligned} R'\boldsymbol{A}'^{+}_1\boldsymbol{A}'^{+}_2 &= \boldsymbol{A}'^{+}_2\boldsymbol{A}'^{+}_1\mathcal{R}', \qquad R'\boldsymbol{A}'_2\boldsymbol{A}'_1 = \boldsymbol{A}'_1\boldsymbol{A}'_2\mathcal{R}', \\ \boldsymbol{A}'_2\boldsymbol{A}'^{+}_1 &= \boldsymbol{I}_{21} + R'^{t_1}\mathcal{R}'^{t_1}\boldsymbol{A}'^{+}_1\boldsymbol{A}'_2, \end{aligned} \tag{3}$$

or

$$\begin{aligned} R'\boldsymbol{A}'^{+}_1\boldsymbol{A}'^{+}_2 &= \boldsymbol{A}'^{+}_2\boldsymbol{A}'^{+}_1\mathcal{R}', \qquad R'\tilde{\boldsymbol{A}}'_1\tilde{\boldsymbol{A}}'_2 = \tilde{\boldsymbol{A}}'_2\tilde{\boldsymbol{A}}'_1\mathcal{R}', \\ \tilde{\boldsymbol{A}}'_2\tilde{\boldsymbol{A}}'^{+}_1 &= \boldsymbol{C}'_{12} + q^2\boldsymbol{A}'^{+}_1\tilde{\boldsymbol{A}}'_2\tilde{R}'^{-1}\tilde{\mathcal{R}}'^{-1}, \end{aligned} \tag{4}$$

where t_1 (resp. t_2) denotes transposition in the first (resp. second) space of the tensor product, $\tilde{R}'$ is defined by $\tilde{R}' = qC'_1\left(R'^{-1}\right)^{t_1}C'^{-1}_1 = qC'_2\left(R'^{t_2}\right)^{-1}C'^{-1}_2$, and similar relations hold for $\tilde{\mathcal{R}}'$. The transformations leaving Eqs. (3) and (4) invariant are $\varphi'\left(\boldsymbol{A}'^{+}\right) = \boldsymbol{A}'^{+}T'\mathcal{T}'$, $\varphi'(\boldsymbol{A}') = T'^{-1}\mathcal{T}'^{-1}\boldsymbol{A}'$, and $\varphi'\left(\boldsymbol{A}'^{+}\right) = \boldsymbol{A}'^{+}T'\mathcal{T}'$, $\varphi'(\tilde{\boldsymbol{A}}') = \tilde{\boldsymbol{A}}'\tilde{T}'\tilde{\mathcal{T}}'$, respectively. Here $\tilde{T}'$ and $\tilde{\mathcal{T}}'$ are defined by $\tilde{T}' = C'^{-1}\left(T'^{-1}\right)^t C'$, and $\tilde{\mathcal{T}}' = \mathcal{C}'^{-1}\left(\mathcal{T}'^{-1}\right)^t\mathcal{C}'$.

There exists another independent set of $\mathrm{GL}_q(n) \times \mathrm{GL}_q(m)$-covariant q-bosonic operators, which satisfy equations similar to Eq. (3) or (4), but with $R'_{12} \to R'^{-1}_{21}$, $\mathcal{R}'_{12} \to \mathcal{R}'^{-1}_{21}$, implying $q^{-1}\tilde{R}'_{12} \to q\tilde{R}'^{-1}_{21}$, $q^{-1}\tilde{\mathcal{R}}'_{12} \to q\tilde{\mathcal{R}}'^{-1}_{21}$.

4 $\mathrm{GL}_h(n) \times \mathrm{GL}_h(m)$-covariant h-bosonic algebras

Let us apply the contraction procedure of Sec. 2 to the $\mathrm{GL}_q(n) \times \mathrm{GL}_q(m)$-covariant q-bosonic algebras, given in two equivalent forms in Eqs. (3) and (4), respectively. Since we now have

two copies of $\mathrm{GL}_q(N)$, we have to consider two transformation matrices $g = \sum_i e_{ii} + \eta e_{1n}$, and $\mathfrak{g} = \sum_s e_{ss} + \eta e_{1m}$, acting on $\mathrm{GL}_q(n)$ and $\mathrm{GL}_q(m)$, respectively.

Let us first consider Eq. (3), and introduce transformed q-bosonic operators defined by $\boldsymbol{A}''^{+} = \boldsymbol{A}'^{+}\boldsymbol{g}$, $\boldsymbol{A}'' = \boldsymbol{g}^{-1}\boldsymbol{A}'$, where $\boldsymbol{g} = g\,\mathfrak{g}$. By using the property $R'^{t}_{12} = R'_{21}$, and a similar one for $\mathcal{R}'$, it is straightforward to show that Eq. (3) becomes

$$\begin{aligned} \boldsymbol{A}''^{+}_1 \boldsymbol{A}''^{+}_2 &= \boldsymbol{A}''^{+}_2 \boldsymbol{A}''^{+}_1 R''^{-1}_{21} \mathcal{R}''_{12}, \qquad \boldsymbol{A}''_1 \boldsymbol{A}''_2 = R''_{12} \mathcal{R}''^{-1}_{21} \boldsymbol{A}''_2 \boldsymbol{A}''_1, \\ \boldsymbol{A}''_2 \boldsymbol{A}''^{+}_1 &= \boldsymbol{I}_{21} + R''^{t_1} \mathcal{R}''^{t_1} \boldsymbol{A}''^{+}_1 \boldsymbol{A}''_2. \end{aligned} \tag{5}$$

Since R and $\mathcal{R}$ are triangular, in the $q \to 1$ limit the h-bosonic operators $\boldsymbol{A}^{+}_{is} = \lim_{q\to 1} \boldsymbol{A}''^{+}_{is}$, $\boldsymbol{A}_{is} = \lim_{q\to 1} \boldsymbol{A}''_{is}$ satisfy the relations

$$\begin{aligned} \boldsymbol{A}^{+}_1 \boldsymbol{A}^{+}_2 &= \boldsymbol{A}^{+}_2 \boldsymbol{A}^{+}_1 R\mathcal{R}, \qquad \boldsymbol{A}_1 \boldsymbol{A}_2 = R\mathcal{R} \boldsymbol{A}_2 \boldsymbol{A}_1, \\ \boldsymbol{A}_2 \boldsymbol{A}^{+}_1 &= \boldsymbol{I}_{21} + R^{t_1} \mathcal{R}^{t_1} \boldsymbol{A}^{+}_1 \boldsymbol{A}_2, \end{aligned} \tag{6}$$

defining a $\mathrm{GL}_h(n) \times \mathrm{GL}_h(m)$-comodule algebra. The transformation $\varphi\left(\boldsymbol{A}^{+}\right) = \boldsymbol{A}^{+} T \mathcal{T}$, $\varphi(\boldsymbol{A}) = T^{-1}\mathcal{T}^{-1}\boldsymbol{A}$, where $T_{ij} \in \mathrm{GL}_h(n)$, $\mathcal{T}_{st} \in \mathrm{GL}_h(m)$, leaves Eq. (6) invariant.

Three properties of Eq. (6) are worth noting: (1) Had we started instead from the second form of Eq. (3) corresponding to the substitutions $R'_{12} \to R'^{-1}_{21}$, $\mathcal{R}'_{12} \to \mathcal{R}'^{-1}_{21}$, we would have obtained the same contraction limit (6), owing to the triangularity of R and $\mathcal{R}$. (2) Contrary to what happens in the q-bosonic case, $\boldsymbol{A}_{is}$ can never be considered as the adjoint of $\boldsymbol{A}^{+}_{is}$, since no *-structure is known on $\mathrm{GL}_h(N)$. (3) For $m = 1$, Eq. (6) is consistent with the general form of $\mathcal{H}$-covariant deformed bosonic algebras for triangular $\mathcal{H}$, obtained by Fiore [12].

Let us next consider Eq. (4), and define $\boldsymbol{A}''^{+} = \boldsymbol{A}'^{+}\boldsymbol{g}$, $\tilde{\boldsymbol{A}}'' = \tilde{\boldsymbol{A}}'\boldsymbol{g}$, where $\boldsymbol{g}$ is the same as before. Compatibility of the $\tilde{\boldsymbol{A}}''$ and $\boldsymbol{A}''$ definitions with $\tilde{\boldsymbol{A}}'' = \boldsymbol{A}''\boldsymbol{C}''$, where $\boldsymbol{C}'' = C''\mathcal{C}''$, leads to $C'' = g^t C' g$, $\mathcal{C}'' = \mathfrak{g}^t \mathcal{C}' \mathfrak{g}$. A simple calculation shows that for $n > 1$, a contraction limit of C'' only exists for even n values, and is given by $C = \lim_{q\to 1} C'' = \sum_i (-1)^i e_{ii'} + (n-1) h e_{nn}$. Similar results hold for $\mathcal{C} = \lim_{q\to 1} \mathcal{C}''$.

Restricting the range of n, m values to $\{1, 2, 4, 6, \ldots\}$, we obtain that after transformation, Eq. (4) contracts into

$$\begin{aligned} \boldsymbol{A}^{+}_1 \boldsymbol{A}^{+}_2 &= \boldsymbol{A}^{+}_2 \boldsymbol{A}^{+}_1 R\mathcal{R}, \qquad \tilde{\boldsymbol{A}}_1 \tilde{\boldsymbol{A}}_2 = \tilde{\boldsymbol{A}}_2 \tilde{\boldsymbol{A}}_1 R\mathcal{R}, \\ \tilde{\boldsymbol{A}}_2 \boldsymbol{A}^{+}_1 &= \boldsymbol{C}_{12} + \boldsymbol{A}^{+}_1 \tilde{\boldsymbol{A}}_2 \tilde{R}^{-1} \tilde{\mathcal{R}}^{-1}, \end{aligned} \tag{7}$$

where $\boldsymbol{C} = C\mathcal{C}$, $\tilde{R} = \lim_{q\to 1} \left(g^{-1} \otimes g^{-1}\right) \tilde{R}'(g \otimes g) = C_1^{-1} \left(R^{-1}\right)^{t_1} C_1 = C_2^{-1} \left(R^{t_2}\right)^{-1} C_2$, and similarly for $\tilde{\mathcal{R}}$. For such restricted n, m values, Eq. (7) yields another form of the $\mathrm{GL}_h(n) \times \mathrm{GL}_h(m)$-covariant h-bosonic algebra defined in Eq. (6) for arbitrary n, m values. The transformation leaving Eq. (7) invariant is $\varphi\left(\boldsymbol{A}^{+}\right) = \boldsymbol{A}^{+} T \mathcal{T}$, $\varphi(\tilde{\boldsymbol{A}}) = \tilde{\boldsymbol{A}} \tilde{T} \tilde{\mathcal{T}}$, where $\tilde{T} = C^{-1} \left(T^{-1}\right)^t C$, $\tilde{\mathcal{T}} = \mathcal{C}^{-1} \left(\mathcal{T}^{-1}\right)^t \mathcal{C}$. However, for n and/or $m \in \{3, 5, 7, \ldots\}$, the contraction procedure does not preserve the equivalence between Eqs. (3) and (4), since only the former has a limit.

5 $\mathbf{GL}_h(2)$ and $\mathbf{GL}_h(2) \times \mathbf{GL}_h(2)$-covariant h-bosonic algebras

For $n = 2$, $m = 1$, by making the substitutions

$$R = \begin{pmatrix} 1 & h & -h & h^2 \\ 0 & 1 & 0 & h \\ 0 & 0 & 1 & -h \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & -1 \\ 1 & h \end{pmatrix}, \qquad \mathcal{R} = \mathcal{C} = 1, \tag{8}$$

into Eqs. (6) and (7), we obtain that A_1^+, A_2^+, A_1, A_2 satisfy the commutation relations

$$\begin{aligned} \left[A_1^+, A_2^+\right] &= h\left(A_1^+\right)^2, \qquad [A_1, A_2] = hA_2^2, \\ \left[A_2, A_1^+\right] &= 0, \qquad \left[A_1, A_2^+\right] = h\left(-A_1^+ A_1 - A_2^+ A_2 + hA_1^+ A_2\right), \\ \left[A_1, A_1^+\right] &= \left[A_2, A_2^+\right] = I + hA_1^+ A_2, \end{aligned} \tag{9}$$

while A_1^+, A_2^+, $\tilde{A}_1$, $\tilde{A}_2$ fulfil

$$\begin{aligned} \left[A_1^+, A_2^+\right] &= h\left(A_1^+\right)^2, \qquad [\tilde{A}_1, \tilde{A}_2] = h\tilde{A}_1^2, \\ [\tilde{A}_1, A_1^+] &= 0, \qquad [\tilde{A}_2, A_2^+] = h(I - A_1^+ \tilde{A}_2 + A_2^+ \tilde{A}_1 + hA_1^+ \tilde{A}_1), \\ [\tilde{A}_1, A_2^+] &= -[\tilde{A}_2, A_1^+] = I + hA_1^+ \tilde{A}_1. \end{aligned} \tag{10}$$

Both sets of operators $\left(A_1^+, A_2^+\right)$ and $(\tilde{A}_1, \tilde{A}_2)$ may be considered as the components $m = 1/2$ and $m = -1/2$ of ITO of rank $1/2$, or spinors, with respect to the quantum algebra $\mathrm{U}_h(\mathrm{sl}(2))$. By considering the adjoint action of the $\mathrm{U}_h(\mathrm{sl}(2))$ generators on such spinors, Aizawa [8] recently realized them in terms of standard bosonic operators a_1^+, a_2^+, a_1, a_2,

$$\begin{aligned} A_1^+ &= \left(1 - \tfrac{h}{2} J_+\right)^{-1} a_1^+, \qquad A_2^+ = \left(1 - \tfrac{h}{2} J_+\right) a_2^+ + \tfrac{h}{2}\left(A_1^+ - 2a_1^+ J_0\right), \\ \tilde{A}_1 &= \left(1 - \tfrac{h}{2} J_+\right)^{-1} a_2, \qquad \tilde{A}_2 = -\left(1 - \tfrac{h}{2} J_+\right) a_1 + \tfrac{h}{2}\left(\tilde{A}_1 - 2a_2 J_0\right), \end{aligned} \tag{11}$$

where $J_+ = a_1^+ a_2$, and $J_0 = \left(a_1^+ a_1 - a_2^+ a_2\right)/2$ are sl(2) generators. As can be easily checked, the operators (11) satisfy Eq. (10), as it should be.

Equation (10) can be recast into an alternative form by using coupled commutators

$$\left[U^{j_1}, V^{j_2}\right]^j_m \equiv \left[U^{j_1} \times V^{j_2}\right]^j_m - (-1)^\epsilon \left[V^{j_2} \times U^{j_1}\right]^j_m, \tag{12}$$

where U^{j_1} and V^{j_2} denote two ITO of rank j_1 and j_2 with respect to $\mathrm{U}_h(\mathrm{sl}(2))$, respectively, $\epsilon = j_1 + j_2 - j$,

$$\left[U^{j_1} \times V^{j_2}\right]^j_m \equiv \sum_{m_1 m_2} \langle j_1 m_1, j_2 m_2 | jm \rangle_h \, U^{j_1}_{m_1} V^{j_2}_{m_2}, \tag{13}$$

and $\langle\, ,\, |\, \rangle_h$ denotes a $\mathrm{U}_h(\mathrm{sl}(2))$ CGC, as determined in Ref. [7]. The results read

$$[A^+, A^+]^0_0 = [\tilde{A}, \tilde{A}]^0_0 = [\tilde{A}, A^+]^1_m = 0, \qquad [\tilde{A}, A^+]^0_0 = \sqrt{2}\, I. \tag{14}$$

For $n = m = 2$, $\mathcal{R}$ and $\mathcal{C}$ take the same form as R and C in Eq. (8). Relations similar to those in Eqs. (9) and (10) can be easily written. The operators $\boldsymbol{A}^+_{is}$, $\tilde{\boldsymbol{A}}_{is}$, i, $s = 1, 2$, may now be considered as the components of double spinors with respect to $\mathrm{U}_h(\mathrm{sl}(2)) \times \mathrm{U}_h(\mathrm{sl}(2))$, and they satisfy the coupled commutation relations

$$\begin{aligned} [\boldsymbol{A}^+, \boldsymbol{A}^+]^{1,0}_{m,0} &= [\boldsymbol{A}^+, \boldsymbol{A}^+]^{0,1}_{0,m'} = [\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{A}}]^{1,0}_{m,0} = [\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{A}}]^{0,1}_{0,m'} = 0, \\ [\tilde{\boldsymbol{A}}, \boldsymbol{A}^+]^{j,j'}_{m,m'} &= 2\delta_{j,0}\delta_{j',0}\delta_{m,0}\delta_{m',0}\boldsymbol{I}, \end{aligned} \tag{15}$$

where in the definition of coupled commutators there now appear two ϵ phases, and two $\mathrm{U}_h(\mathrm{sl}(2))$ CGC.

It is remarkable that both Eqs. (14) and (15) are formally identical with those for sl(2) and $\mathrm{sl}(2) \times \mathrm{sl}(2)$, respectively. Contrary to what happens in the q-bosonic case where the commutators are q-deformed, here all the dependence upon the deforming parameter h is contained in the CGC.

6 Conclusion

In this communication, we showed that $\mathrm{GL}_h(n) \times \mathrm{GL}_h(m)$-covariant h-bosonic algebras can be obtained by contracting $\mathrm{GL}_q(n) \times \mathrm{GL}_q(m)$-covariant q-bosonic ones. Some extensions of the present work to h-fermionic and multiparametric algebras are under current investigation.

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